Original Research

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Water Retention Models for Scale-Variant and Scale-Invariant Drainage of Mass Prefractal Porous Media

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This research describes the development of water retention models incorporating the effects of partial drainage in random mass prefractal porous media. The pore-size distribution, as well as the connectivity of pores, determines the drained pore volume as a function of suction. The concept of probability of drainage leads to a general scale-variant drainage model (GM) in which the proportion of pores that drain at a given suction level is dependent on the fractal dimension of the drained pore phase, D_d, and the proportion of pores that drain at the first suction level or air entry value. Two simplified cases of the general model are also presented. The first simplified model (simplified case 1 [SC1]) is a special case of the GM in which all of the largest pores drain completely at the first suction level. The second model (simplified case 2 [SC2]) is a scale-invariant model in which the proportion of drained pores for each suction level remains constant and is obtained by setting D_d equal to the mass fractal dimension, D of the porous medium. Fitting each model to numerically simulated drainage curves for random two-dimensional prefractal porous media with known D values shows that the GM fitted the numerical data much better than either the SC1 or SC2 models, which were less flexible at high D values. Estimates of D_d for the GM and SC1 models approached D when D was less than the critical value for percolation, that is, $D_c \sim 1.716$. Independent estimates of the probability of drainage indicate that the connectivity of water-filled pores decreases as a result of the lower porosities associated with higher D values. A novel experimental protocol is suggested for testing these theoretical observations.

Abbreviations: AIC, Akaike's information criterion; ESS, error sum of squares; GM, general scale-variant drainage model; SC1, simplified case 1; SC2, simplified case 2.

The use of fractal methods for quantifying soil hydraulic functions is a powerful tool to understand the flow of fluids and contaminants in the unsaturated zone. Fractal models are based on physical parameters that lead to much easier interpretation compared to empirical models. Fractals are iterative geometrical models for describing irregular and fragmented systems. As such, they are ideally suited to simulate the hierarchical and heterogeneous nature of soil structure.

Models based on fractal geometry are being used increasingly to derive physically based expressions for soil hydraulic properties, particularly the saturation–capillary pressure curve (Giménez et al., 1997; Bird et al., 2000; Wang et al., 2005). One of the earliest and most widely accepted fractal water retention models was derived by Rieu and Sposito (1991). This model does not take into account the randomness of natural porous media and incomplete connectivity of individual pores that may result in partial drainage of pores. Numerical capillary drainage simulations in random fractal structures showed that a lack of pore connectivity in the Rieu and Sposito (1991) model caused deviations between predicted and observed data (Perrier et al., 1995; Bird and Dexter, 1997).

Perrier et al. (1999) proposed a pore–solid fractal model in which the initiator includes pores and solids, which have constant fractions, as well as an iterative phase space. The iterative phase vanishes as the iteration process approaches infinity. Bird et al. (2000) presented a water retention function based on the pore–solid fractal approach. Wang et al. (2005) tested Bird et al.'s (2000) model against a very large data set for different types of soils by fitting the model to experimental data. Their results indicate that Bird et al.'s (2000) function provides a better fit than the Rieu and Sposito (1991) and Brooks and Corey (1964) equations, which were shown to be simplified cases of the pore–solid fractal model.

Fractal water retention equations, including Bird et al.'s (2000) model, are often used to estimate the fractal dimension of the porous medium. During drainage of a fractal porous medium, both the fractal dimension and the connectivity of pores determine the drained pore volume as function of suction. However, since the above-mentioned models do not incorporate the effect of pore connectivity explicitly, estimates of the fractal dimension obtained by fitting these models to experimental data may not be accurate. Rather, they should be thought of as apparent fractal dimensions.

Perfect (2005) presented a water retention curve model introducing the concept of scale-invariant probability of drain-

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age to account for incomplete pore drainage during monotonic drying of a random prefractal porous medium. The probability of drainage, P_d, defined as the probability of pores of length (l) emptying during drainage, was stated to be scale invariant. Although the final saturation–capillary pressure equation given by Eq. [15] in Perfect (2005) turns out to be correct, the derivation given in Perfect (2005) leads to a scale-variant probability in which the ratio of the number of drained pores to number of total pores changes at each iteration level of drainage. Thus, the conclusions drawn about the scale-invariant P_d parameter are not correct, and scale-invariant pore drainage can only be obtained as a simplified case of a more general model, which is provided below.

The purpose of this article is to correct the error in the previous study by Perfect (2005) and to further explore modeling of partial drainage in a random mass prefractal porous medium. We present soil water retention models for various cases of interest based on scale-variant and scale-invariant conceptualizations of incomplete pore drainage. The different assumptions involved are tested by fitting each model to the numerically simulated monotonic drainage curves for the random two-dimensional prefractal porous media investigated by Sukop et al. (2001). Probabilities of drainage are also estimated from the drainage curves for comparison with the model assumptions.

Theory

Simplified soil water retention models for scale-variant and scale-invariant pore drainage will be derived as special cases of a more general model, which is presented first.

General Fractal Drainage Model

The numbers of solids, N_s, and pores, N_p, of length l in a mass prefractal porous medium are given by the following:

$$N_s(l) = l^{-D} = \left(b^i\right)^D \qquad [1]$$

$$N_p(l) = n_p b^{(i-1)D} = \left(b^E - b^D\right) b^{(i-1)D} \qquad [2]$$

where i is the iteration level, E is the Euclidean dimension, D is the mass fractal dimension defined as $\log[N_s(l)/N_s(bl)]/\log b$, b is the scale factor, and n_p is the number of pores in the generator. As drying occurs, not all pores of a given size drain at the appropriate suction because of incomplete pore connectivity. The number of drained pores, N_d, is assumed to be fractal and proportional to a power of the length l as expressed by

$$N_d(l) = P\left(b^E - b^D\right) b^{(i-1)D_d} \qquad [3]$$

where P is the ratio of the drained pore space to the total pore space in the generator, $0 \leq P \leq 1$, and D_d is the fractal dimension for the drained pore space, which can be defined as

$$D_d = \frac{\log\left[N_d(l)/N_d(bl)\right]}{\log b} \leq D \qquad [4]$$

The cumulative volume of the drained pore space, $V_d(l)$, can be calculated from Eq. [3] using the expression

$$V_d(l) = \sum_{n=1}^{i} \frac{N_d(l)}{b^{En}} = P\left(\frac{b^E - b^D}{b^E - b^{D_d}}\right)\left[1 - \left(\frac{1}{b^i}\right)^{E-D_d}\right] \qquad [5]$$

where $1 \leq n \leq i$ is the nth iteration level. Then, the volumetric water content of the partially drained prefractal porous medium is given by:

$$\theta = \phi - V_d(l) = \phi - P\left(\frac{b^E - b^D}{b^E - b^{D_d}}\right)\left[1 - \left(\frac{1}{b^i}\right)^{E-D_d}\right] \qquad [6]$$

Expressed in terms of relative saturation, S, Eq. [6] becomes

$$S = 1 - \frac{P}{\phi}\left(\frac{b^E - b^D}{b^E - b^{D_d}}\right)\left[1 - \left(\frac{1}{b^i}\right)^{E-D_d}\right] \qquad [7]$$

Invoking the Young–Laplace expression (de Gennes et al., 2004), $1/b^i$ in Eq. [7] can be replaced with the normalized capillary pressure, h_{min}/h, where h is the capillary pressure and h_{min} is the minimum capillary pressure that drains the largest pores, giving

$$S = 1 - \frac{P}{\phi}\left(\frac{b^E - b^D}{b^E - b^{D_d}}\right)\left[1 - \left(\frac{h}{h_{min}}\right)^{D_d-E}\right] \qquad [8]$$

which is identical to Eq. [15] in Perfect (2005). Equation [8] reduces to the Rieu and Sposito (1991) model when $P = 1$ and $D_d = D$.

An example realization of drainage for the general scale-variant drainage model (GM) with $P = 0.5$ and $D_d = 1.630\ldots$ in a random Sierpinski carpet of unit length constructed using $E = 2$, $b = 3$, $j = 2$ (the last iteration level of the carpet), and $D = 1.771\ldots$ is presented in Fig. 1a. Since $P = 0.5$, only one of the largest pores of size of $1/b$ drains. At the second iteration level, $i = 2$, there are $N_p(1/b^2)$ pores and the proportion of those pores that drain is given by Pb^{D_d-D}; that is, 6 out of 14 pores of size $1/b^2$ become empty (Fig. 1a).

The sensitivity of Eq. [8] to the parameters P and D_d is demonstrated for a three-dimensional mass prefractal porous medium or Menger sponge in Fig. 2. Figure 2a shows changes in S as function of $\log_b(h/h_{min})$ while P is varied from 0.1 to 1 and D_d is kept constant at 2.5 ($<D = 2.680\ldots$). When P is equal to unity, all the largest pores drain. The slope of the saturation versus normalized capillary pressure curve decreases as P increases. Likewise, Fig. 2b shows that S decreases as D_d increases from 2 to D. S is most sensitive to D_d at values approaching D, and D_d has little impact on S near the wet end of the water retention curve.

Simplified Case 1 (SC1): Scale Variant

If the probability of drainage in the generator, P, is assumed to be unity such that all the largest pores drain completely, Eq. [3] becomes

$$N_d(l) = \left(b^E - b^D\right) b^{(i-1)D_d} \qquad [9]$$

and the relative saturation is now given by

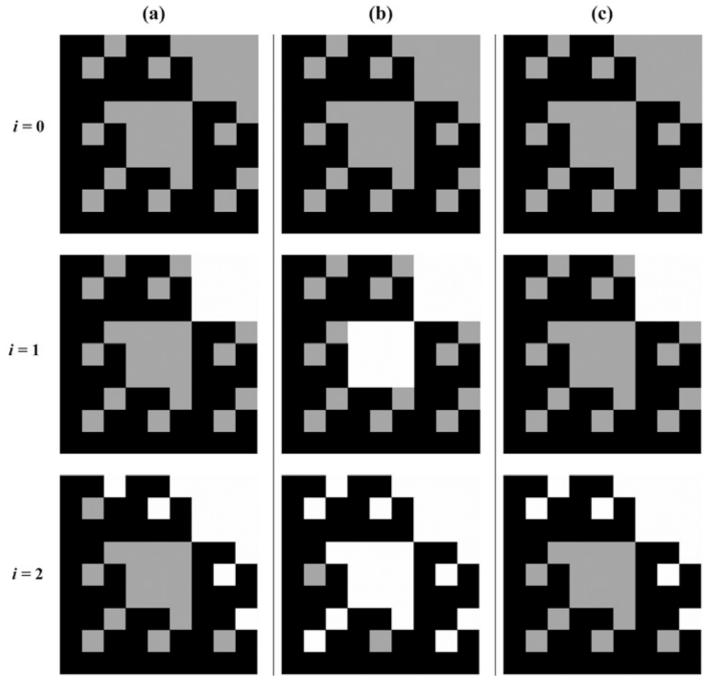

FIG. 1. Example realizations for (a) general scale-variant drainage model, (b) simplified case 1, and (c) simplified case 2 in a b = 3, j = 2, D = 1.771... random Sierpinski carpet with D_d = 1.630..., and P = 0.5 (black = solid, white = air-filled pore, and gray = water-filled pore). (b = scale factor; j = the last iteration level; D = mass fractal dimension; D_d = fractal dimension for the drained pore space; P = ratio of the drained pore space to the total pore space in the generator.)

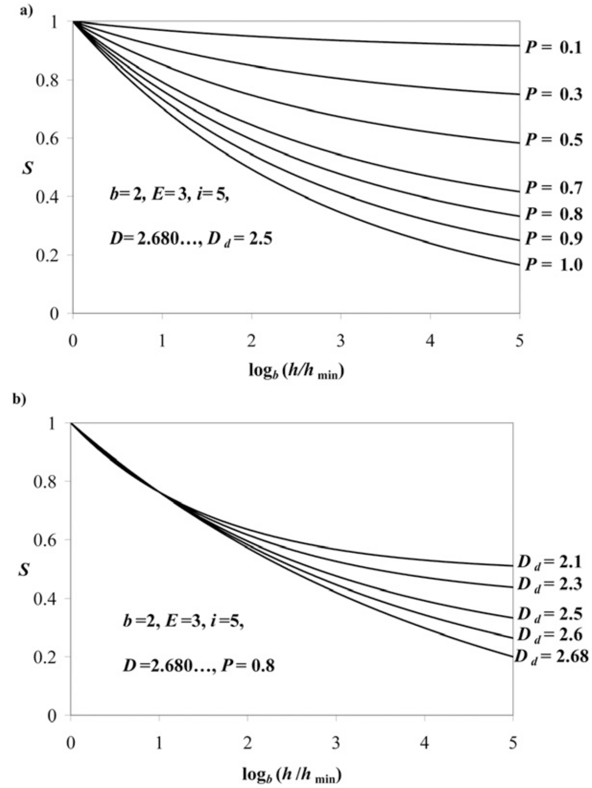

FIG. 2. Sensitivity of the general scale-variant drainage model (GM) to the parameters P and D_d. (S = relative saturation; b = scale factor; E = Euclidean dimension; i = iteration level; D = mass fractal dimension; D_d = fractal dimension for the drained pore space; h/h_{min} = normalized capillary pressure; P = ratio of the drained pore space to the total pore space in the generator.)

$$S = 1 - \frac{1}{\phi}\frac{\left(b^{E} - b^{D}\right)}{\left(b^{E} - b^{D_d}\right)}\left[1 - \left(\frac{h}{h_{min}}\right)^{D_d - E}\right] \qquad [10]$$

Figure 1b presents an example of drainage for the simplified scale-variant model (SC1) with D_d = 1.630…. Since P = 1, both of the largest pores drain at first iteration level, while at the second iteration level, a $b^{D_d - D}$ fraction of the $N_p(1/b^2)$ pores drains; that is, 12 out of 14 pores sized $1/b^2$ become empty (Fig. 1b).

Simplified Case 2 (SC2): Scale Invariant

When $D_d = D$, the ratio of the number of drained pores to total pores at any iteration level i remains constant and equal to P. In this case, Eq. [3] is written as

$$N_d(l) = P\left(b^{E} - b^{D}\right)b^{(i-1)D} \qquad [11]$$

yielding the following relative saturation function:

$$S = 1 - \frac{P}{\phi}\left[1 - \left(\frac{h}{h_{min}}\right)^{D-E}\right] \qquad [12]$$

This equation has the same form as the water retention model for a pore–solid fractal proposed by Bird et al. (2000), although the interpretation of the model parameters is different.

Figure 1c presents an example of drainage for the scale-invariant model (SC2) with P = 0.5…. At each iteration level, the ratio of the empty pores to filled pores is constant; that is, at the first iteration level, one of the two pores of size $1/b$ drains, while at i = 2, 7 out of the 14 pores of size $1/b^2$ drain (Fig. 1c).

Probability of Drainage

Perfect (2005) defined the probability of drainage at any level n in the fractal hierarchy as the ratio of the number of drained pores to the total number of pores of length l, which was expressed as

$$P_d(l) = \frac{N_d(l)}{N_p(l)} \qquad [13]$$

It was then assumed that the same proportions of pores empty at each iteration level; that is, $P_d(l) = P_d(bl)$. However, this approach does not account for the continued drainage of pores of length l at subsequent suction levels or any imbibition of drainage water from previously nondrained pores of length $\geq bl$. In reality, the relationship between $P_d(l)$ and $P_d(bl)$ is much more complicated, and the cumulative effect of all inputs and outputs of water must be taken into account for each pore size class.

To further investigate the pore-scale drainage processes underlying the different analytical models in the GM, we present a new probabilistic expression for N_d that incorporates the effect of connectivity among pores with different sizes and allows continuing drainage of pores of sizes $1/b^{i-1}$, $1/b^{i-2}$, and so on, into pores of size $1/b^i$ at different suctions. A more appropriate form of the probability of drainage will also be formulated.

Depending on the geometrical arrangement (*lacunarity*) of the prefractal porous medium, any pores of length $1/b$ that do not drain at the appropriate suction may remain full or may empty into pores of length $1/b^2$, $1/b^3$… or $1/b^i$ as $h \to \infty$. Similarly, nondraining pores of length $1/b$ may never empty or may later drain into pores of length $1/b^2$, $1/b^3$… or $1/b^i$ (Perfect, 2005). Drainage of the remaining water-filled pores from the first iteration level may continue at subsequent iteration levels.

At iteration level 1, which corresponds to the air-entry value or minimum suction, a P_1 fraction times the volume of pores generated at $i = 1$ drains; that is $P_1 N_p^{(1)} / b^E$. At iteration level 2, a P_2 fraction of the pore volume generated at $i = 2$ plus the remaining pore volume from iteration level 1 drains; that is, $P_2[N_p^{(2)} / b^{2E} + (1-P_1)N_p^{(1)} / b^E]$. Then, after two iterations, the total volume of water remaining in pores of sizes $1/b$ and $1/b^2$ can be calculated as

$$\left(1-P_2\right)\left[\frac{N_p^{(2)}}{b^{2E}} + \left(1-P_1\right)\frac{N_p^{(1)}}{b^E}\right] = \left(1-P_2\right)\frac{N_p^{(2)}}{b^{2E}} + \left(1-P_2\right)\left(1-P_1\right)\frac{N_p^{(1)}}{b^E} \qquad [14]$$

At iteration level 3, the drained pore volume is the summation of P_3 times the volume of pores generated at $i = 3$ and the remaining pore volume from iteration level 2. This conceptual model for the drainage process can be formulated as

$$n=1 \quad \phi - \theta_1 = P_1 \frac{N_p^{(1)}}{b^E}$$

$$n=2 \quad \theta_2 - \theta_1 = P_2\left[\frac{N_p^{(2)}}{b^{2E}} + \left(1-P_1\right)\frac{N_p^{(1)}}{b^E}\right]$$

$$n=3 \quad \theta_3 - \theta_2 = P_3\left[\frac{N_p^{(3)}}{b^{3E}} + \left(1-P_2\right)\frac{N_p^{(2)}}{b^{2E}} + \left(1-P_2\right)\left(1-P_1\right)\frac{N_p^{(1)}}{b^E}\right] \qquad [15]$$

$$\vdots \qquad \vdots$$

$$n=i \quad \theta_i - \theta_{i-1} = P_i\left[\frac{N_p^{(i)}}{b^{iE}} + \left(1-P_{i-1}\right)\frac{N_p^{(i-1)}}{b^{(i-1)E}} + \cdots + \left(1-P_{i-1}\right)\left(1-P_{i-2}\right)\ldots\left(1-P_1\right)\frac{N_p^{(1)}}{b^E}\right]$$

where θ_1, θ_2, θ_3,… are the volumetric water contents corresponding to the suction levels $n = 1, 2, 3$… The probability of drainage of the remaining pore volume at any i or any corresponding suction, P_i, can now be expressed as

$$P_i = \frac{\theta_{i-1} - \theta_i}{\sum_{n=1}^{i} \frac{N_p^{(n)}}{b^{nE}} \prod_{k=n}^{i-1}\left(1-P_k\right)} \qquad [16]$$

where Π is the product symbol and if $(i-1) < k$, $\Pi_k^{i-1}(1-P_k) = 1$. To compare this approach with Eq. [3], the drained pore volume at iteration number i can be expressed in terms of N_d and b^i such that $\theta_{i-1} - \theta_i = N_d^{(i)}(1/b^{iE})$. Then, by rearranging Eq. [16], $N_d^{(i)}$ ($l = 1/b^i$) can be written as

$$N_d^{(i)} = P_i b^{iE} \sum_{n=1}^{i} \frac{N_p^{(n)}}{b^{nE}} \prod_{k=n}^{i-1}\left(1-P_k\right) \qquad [17]$$

The P_i values in Eq. [16] and [17] contain information about the connectivity of the pore system and are independent of any assumed fractal drainage behavior. Each P_i value is the percentage of the volume of the connected pores filled with water whose sizes are greater than or equal to $1/b^i$. At the appropriate suction level, i, those pores that are connected to the atmosphere will drain. If b, D, and i values are known a priori for a fractal porous medium, then the P_1, P_2, …, P_i values can be estimated inversely from the resulting water retention curve. By inversely solving Eq. [16] against drained pore volume data from the drainage simulations of Sukop et al. (2001), we can tell how P_i changes as a function of iteration level and D. We can also test the implied assumption of power-law scaling in Eq. [3] by comparison with Eq. [17].

Methods

Simulated Water Retention Data

Bird and Dexter (1997) and Sukop et al. (2001) computed moisture suction relations in two-dimensional prefractal pore networks using an invasion percolation algorithm. They simulated drainage in $b = 3$ and $i = 5$ randomized Sierpinski carpets with different D values by allowing three sides of each prefractal structure to be open to the atmosphere while the bottom was connected to a water sink. According to their algorithm, at a given tension level i, all pores of size greater than $1/b^i$ that are filled with water and are connected to the atmosphere by at least one path consisting of pores no smaller than $1/b^i$ drain. The simulations neglect the effect of pore coalescence and assume applicability of the Young–Laplace equation. Ten simulations were run for each set of carpet parameters.

Nonlinear Fitting

Equations [8], [10], and [12] were fitted to the simulated water retention curves using nonlinear regression (Marquardt method) in SAS (SAS Institute, 1999). Both P and D_d were estimated for the GM, while only D_d was estimated for SC1 and P for SC2. All of the fits converged according to the SAS default convergence criterion (SAS Institute, 1999). The balance between goodness-of-fit and parsimony for the different model fits was evaluated using Akaike's information criterion (AIC). The AIC was estimated by (SAS Institute, 1999)

$$\text{AIC} = \nu \ln\left(\frac{\text{ESS}}{\nu}\right) + 2p \qquad [18]$$

where ν is the number of observations, ESS is the error sum of squares, and p is the number of model parameters. The smaller (the more negative) the AIC value, the better the model.

Inverse Estimation of Probability of Drainage

By using the data obtained from the numerical simulations, the P_i values in Eq. [16] can be calculated by explicitly solving Eq. [15] from the known values of water content versus iteration level since b, D, and saturated water content or porosity are known a priori. For example, at iteration level 1, $P_1 = (\phi - \theta_1)/(1 - b^{D-E})$. As an example of this procedure, Table 1 shows the calculation of the P_i values for water retention data from realization no. 1 of a b = 3, j = 5, D = 1.771… carpet. Since N_p is known for these structures, N_d can also be calculated from the resulting estimates of P_i.

TABLE 1. Example calculation of P_i values for simulated water retention data from realization no. 1 of a b = 3, j = 5, D = 1.771…random Sierpinski carpet.

i	$N_p^{(i)}/b^{2i}$†	θ_i†	$\theta_{i-1} - \theta_i$	$\sum_{n=1}^{i} N_p^{(n)}/b^{nE} \prod_{k=n}^{i-1}(1-P_k)$	P_i†
0	0	0.715	–	–	–
1	0.222	0.493	0.222	0.222	P_1 = 1.000
2	0.173	0.419	0.074	0.173	P_2 = 0.428
3	0.134	0.367	0.052	0.233	P_3 = 0.223
4	0.105	0.340	0.027	0.286	P_4 = 0.094
5	0.081	0.313	0.027	0.340	P_5 = 0.079

† $N_p^{(i)}/b^{2i}$ = areal fraction of pores with $1/b^i$ width; θ_i = areal water content at iteration level i; P_i = probability of drainage.

Results and Discussion

The model equations represented by Eq. [8], [10], and [12] were fitted to numerically simulated monotonic drainage curves for 10 realizations of each of three different generators (b = 3 and D = 1.892…, D = 1.771…and D = 1.630…) of random two-dimensional prefractal porous media (Sukop et al., 2001; Perfect, 2005). Since D, b, and h_{min} were known from the simulations, these parameters were specified in the fitting procedure. The GM is represented by Eq. [8] with two unknown parameters: P and D_d. The SC1 is represented by Eq. [10] with one unknown parameter: D_d. The SC2 is represented by Eq. [12] with one unknown parameter: P.

Figure 3 shows the differences in the performance of the three cases for each D value investigated. The examples presented in Fig. 3 were chosen from the realizations that gave the maximum difference in ESS values between the GM and SC1 and between the GM and SC2, respectively. Figure 3a presents the comparison of the GM and SC1 for the realization resulting in the maximum ESS difference when D = 1.630…. The GM and SC1 result in similar predictions, and their ESS values are very close to each other. Likewise, in Fig. 3b, the GM and SC2 almost overlap, giving the same ESS values up to four digits. Figures 3a and 3b show that all of the models gave similar predictions to the numerical data for D = 1.630…. However, for larger D values (Fig. 3c–3f), the simplified scale-variant (SC1) and scale-invariant (SC2) models showed marked deviations, while the GM always fit the numerical data the best. The Rieu and Sposito (1991) model always showed a large deviation from the simulations for all D values investigated.

A summary of the different fits is presented in Table 2. Estimates of D_d from the GM were greater than those from the SC1, and the maximum difference between the estimates of D_d occurred at the highest D value. Both sets of D_d values approached D when D was less than the critical value for percolation, that is, D_c ~1.716… (Perfect, 2005). This suggests that a larger proportion of the pores drain due to higher pore connectivity when $D < D_c$. When $D > D_c$, the D_d estimates were always smaller than D. The mean P values increased with increasing D_d and decreasing D. This trend was

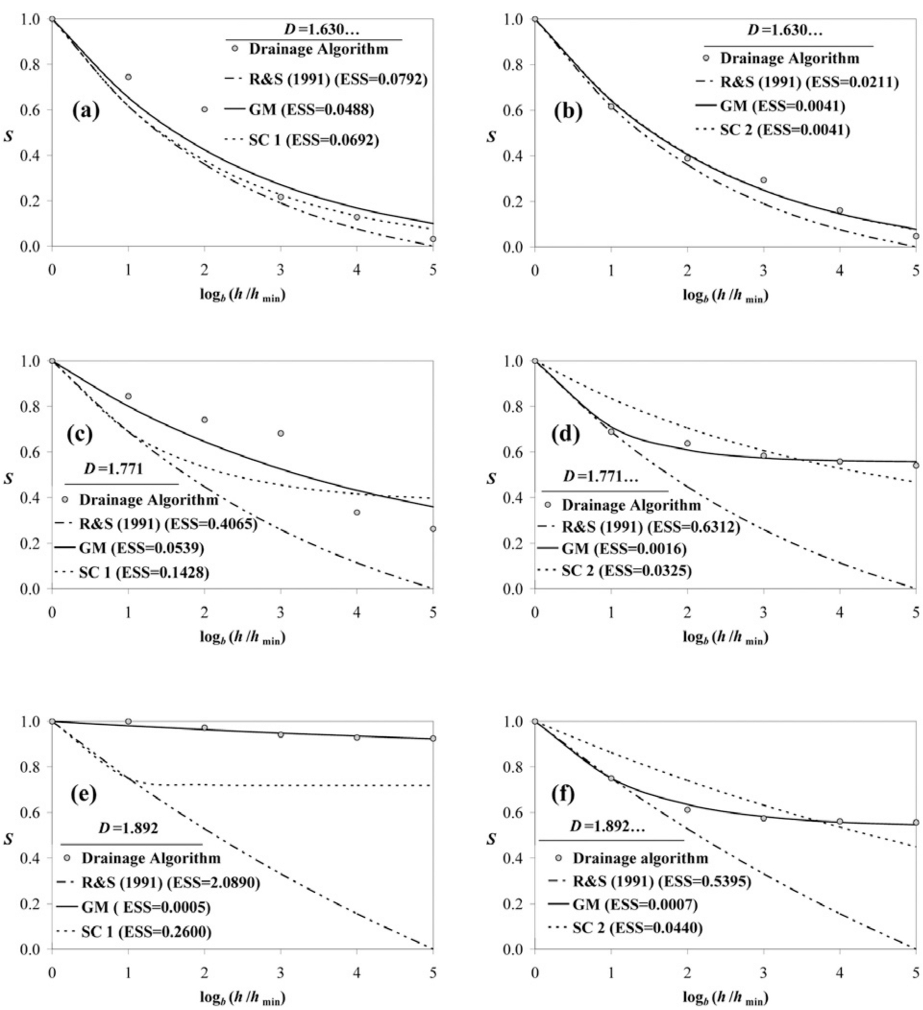

FIG. 3. Comparison of the different water retention models for the realizations giving the maximum difference in error sum of squares (ESS) between the general scale-variant drainage model (GM) and simplified model 1 (SC1) (a, c, and e) and between GM and SC2 (b, d, and f) for D = 1.630…, D = 1.771..., and D = 1.892... (b = 3, j = 5). (S = relative saturation; D = mass fractal dimension; b = scale factor; j = the last iteration level; h/h_{min} = normalized capillary pressure; R&S = Rieu and Sposito.)

more pronounced for the SC2 than the GM, which produced estimates of P much closer to unity.

Overall, the mean ESS for the nonlinear fits ranged from 0.001 to 0.048. As can be seen from Table 2, the GM has the lowest ESS values for all of the D values and also the lowest AIC values for the two largest D values. All of the cases were very similar for the lowest D value investigated. These results indicate that the GM represented by Eq. [8] is the best-fitting model overall. For systems well below the percolation threshold, the SC2 is a viable alternative since it fits just as well as the GM but has one less parameter.

The probabilistic expression developed for the drained pore volume, Eq. [16], was inversely solved for P_i and N_d using the same simulated water retention curves from Sukop et al. (2001). Figure 4 shows how the mean P_i values calculated from 10 realizations for each D value change as function of suction level. The mean P_i for each D investigated generally decreases with suction level except for D = 1.630…, which shows a less pronounced trend, fluctuating between 0.63 and 0.86. Decreasing P_i values with increasing D indicate that the connectivity of water-filled pores decreases as a result of the lower porosity of the randomized two-dimensional carpets with higher mass fractal dimensions. The error bars show one standard deviation around the mean, and their high values indicate a high degree of variability among the 10 different realizations (Fig. 4).

Predicted relationships for N_d obtained by substituting the mean parameter estimates from Table 2 into Eq. [3] compared favorably with the inversely calculated N_d values from Eq. [17] (Fig. 5). This result confirms a power law–type behavior for the drained pore space and the applicability of Eq. [3], at least for random mass prefractal porous media that drain according to the simple invasion percolation algorithm of Bird and Dexter (1997). Further research will be required to evaluate these equations against drainage data simulated using alternative techniques (e.g., lattice Boltzmann) and/or measured on natural porous media. In this context, pore-scale observations of the partitioning of air and water within pores at a given suction level would be particularly useful. An epoxy casting technique developed by Wunderlich (1985) could be used for this purpose. In this technique, air is forced into a soil sample initially saturated with a colored epoxy representing water. After equilibration at a given suction, the epoxy is solidified in situ. Finally, thin sections of the sample are prepared to image the phase distributions.

We have shown in the GM that the assumption D_d = D leads to a theoretical water retention curve, Eq. [12], that is of the same form as the drainage model for a pore–solid fractal proposed by Bird et al. (2000). However, P in the present model represents the scale-invariant probability of drainage, while in Bird et al. (2000), this parameter controls the void: solid ratio at each iteration level, and ultimately the porosity.

While Eq. [15] in Perfect (2005) is correct, it is for general scale-variant drainage rather than scale-invariant drainage. Furthermore, Eq. [7] in Perfect (2005) is incorrect, and there is no simple way of relating D_d to D unless $D << D_c$. This means that estimates of D_d from saturation-capillary pressure data cannot be used to infer values of the underlying mass fractal dimension of the porous medium unless it is assumed that scale-invariant drainage has occurred or that $D << D_c$. Pore-scale experi-

TABLE 2. Mean values of the parameter estimates and goodness-of-fit statistics for the different models fitted to the simulated water retention curves from Sukop et al. (2001) using nonlinear regression. Standard deviations are given in parentheses.

Model type†	D‡	D_d‡	P‡	ESS‡	AIC‡
GM	1.630…	1.630 (0.002)	0.923 (0.017)	0.016 (0.019)	−35.2 (7.5)
	1.771…	1.521 (0.247)	0.819 (0.135)	0.011 (0.017)	−39.8 (8.8)
	1.892…	1.279 (0.367)	0.807 (0.377)	0.001 (0.001)	−55.0 (6.8)
SC1	1.630…	1.568 (0.015)	–	0.024 (0.025)	−34.4 (7.0)
	1.771…	1.292 (0.186)	–	0.032 (0.047)	−36.6 (10.6)
	1.892…	0.876 (0.499)	–	0.048 (0.101)	−49.8 (18.2)
SC2	1.630…	–	0.922 (0.017)	0.018 (0.019)	−37.2 (7.5)
	1.771…	–	0.648 (0.064)	0.022 (0.014)	−33.0 (5.1)
	1.892…	–	0.418 (0.177)	0.032 (0.017)	−32.9 (10.1)

† GM, general scale-variant drainage model; SC1, simplified case 1; SC2, simplified case 2.
‡ D = mass fractal dimension; D_d = fractal dimension for the drained pore space; P = ratio of the drained pore space to the total pore space in the generator; ESS = error sum of squares; AIC = Akaike information criterion.

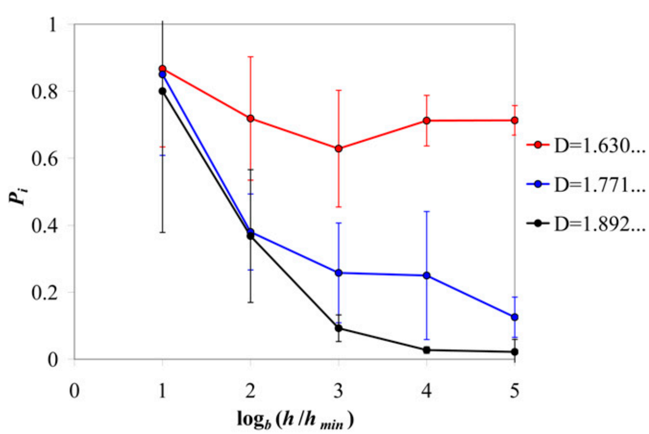

FIG. 4. Mean values of P_i calculated inversely from Eq. [16] as function of suction level. The error bars indicate one standard deviation around the mean. (D = mass fractal dimension; h/h_{min} = normalized capillary pressure; P_i = probability of drainage.)

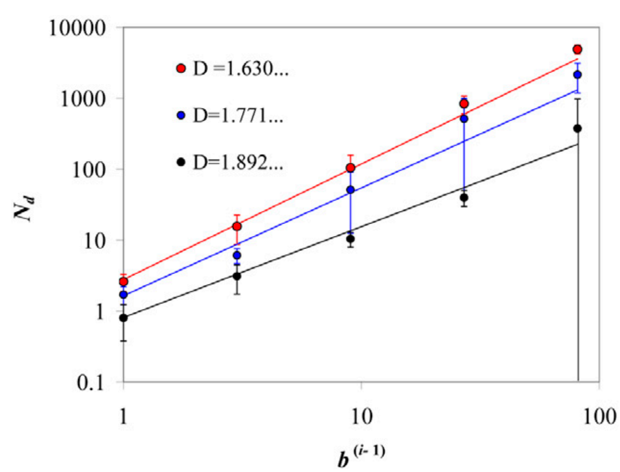

FIG. 5. Comparison of predicted relationships (lines) by substituting mean parameter estimates from Table 2 into Eq. [3] with the observed values of N_d (circles) calculated inversely from the simulated data. The error bars indicate one standard deviation around the mean. (N_d = number of drained pores; D = mass fractal dimension; b = scale factor.)

ments and percolation studies are needed to assess the extent to which these assumptions apply to natural porous media.

By fitting different drainage models to the numerical simulations of Sukop et al. (2001), it was possible to identify the model that best fits saturation-capillary pressure data when D, b, and $h_{\min}$ are known (i.e., the GM). This would not have been possible with experimentally determined water retention curves since these parameters are not known a priori for natural porous media. For soil data sets, Eq. [8], [10], and [12] all take on the same form, which can be written for the purpose of fitting as

$$S = 1 - \frac{\alpha}{\phi}\left[1 - \left(h/h_{\min}\right)^{\beta - E}\right] \qquad [19]$$

where α and β represent the different compound parameters for each case. Perfect (2005) already showed that Eq. [19] provides an excellent fit to water retention curves for six Washington State soils investigated by Campbell and Shiozawa (1992). Based on the present study, we still interpret the estimates of α and β obtained by fitting Eq. [19] to these data as $P\left(b^E - b^D\right)/\left(b^E - b^{D_d}\right)$ and D_d, respectively. However, it is no longer possible to relate these parameters to the pore space geometry of the different soils.

Conclusions

Theoretical water retention equations for a prefractal porous medium have been presented for three cases based on scale-variant and scale-invariant conceptualizations of incomplete pore drainage. The scale-variant drainage models, GM and SC1, allow the proportion of nondraining pores, P_d, to change with pore size and suction level, while in the scale-invariant model, SC2, P_d is a constant. Overall, best estimates of the simulated data were obtained for the GM. For systems well below the percolation threshold, however, the SC2 (which is equivalent to a pore–solid fractal model) is preferred since it fits just as well as the GM and has one less unknown parameter.

We have presented a new probabilistic expression for the drained pore space that incorporates the effect of connectivity among pores with different sizes and allows continuing drainage of pores at different suctions. Extracting an analytical expression for the water content based on this new approach is currently not possible without knowing P_i or $1 - P_i$, or assuming a specific type of distribution as a function of suction level. However, the conceptualization seems promising for future work toward developing a more complete physical model that explicitly includes the effects of both connectivity and fractal dimension. Further work is also required to extend the approaches presented in this study to scale-variant and scale-invariant wetting processes to derive expressions for the main wetting branch and scanning loops of the water retention curve. Experimental studies of partial drainage and wetting at the pore-scale would also be valuable for model validation purposes.

References

Bird, N.R.A., and A.R. Dexter. 1997. Simulation of soil water retention using random fractal networks. Eur. J. Soil Sci. 48:633–641.

Bird, N.R.A., E. Perrier, and M. Rieu. 2000. The water retention function for a model of soil structure with pore and solid fractal distributions. Eur. J. Soil Sci. 51:55–63.

Brooks, R.H., and A.T. Corey. 1964. Hydraulic properties of porous media. Hydrol. Paper 3. Colorado State Univ., Fort Collins.

Campbell, G.S., and S. Shiozawa. 1992. Prediction of hydraulic properties of soils using particle size distribution and bulk density data. p. 317–328. *In* M.Th. van Genuchten, F.J. Leij, and L.J. Lund (ed.) Proc. of the Int. Workshop on Indirect Methods for Estimating the Hydraulic Properties of Unsaturated Soils, Riverside, CA. 11–13 Oct 1989. Univ. of California Press, Berkeley.

de Gennes, P.-G., F. Brochard-Wyart, and D. Quéré. 2004. Capillarity and wetting phenomena: Drops, bubbles, pearls, and waves. Springer-Verlag, New York.

Giménez, D., E. Perfect, W.J. Rawls, and Y. Pachepsky. 1997. Fractal models for predicting soil hydraulic properties: A review. Eng. Geol. 48:161–183.

Perfect, E. 2005. Modeling the primary drainage curve of prefractal porous media. Vadose Zone J. 4:959–966.

Perrier, E., N. Bird, and M. Rieu. 1999. Generalizing the fractal model of soil structure: The pore–solid fractal approach. Geoderma 88:137–164.

Perrier, E., C. Mullon, M. Rieu, and G. d. Marsily. 1995. Computer construction of fractal soil structures: simulation of their hydraulic and shrinkage properties. Water Resour. Res. 31:2927–2943.

Rieu, M., and G. Sposito. 1991. Fractal fragmentation, soil porosity, and soil water properties: 1. Theory. Soil Sci. Soc. Am. J. 55:1231–1238.

SAS Institute. 1999. SAS/STAT user's guide. Version 8. SAS Inst. Cary, NC.

Sukop, M.C., E. Perfect, and N.R.A. Bird. 2001. Water retention of prefractal porous media generated with the homogeneous and heterogeneous algorithms. Water Resour. Res. 37:2631–2636.

Wang, K., R. Zhang, and F. Wang. 2005. Testing the pore-solid fractal model for the soil water retention function. Soil Sci. Soc. Am. J. 69:776–782.

Wunderlich, R.W. 1985. Imaging of wetting and nonwetting phase distributions: Application to centrifuge capillary pressure measurements. p. 1–13. *In* 60th Ann. Tech. Conf. and Exhibit, Las Vegas, NV. 22–25 Sept. 1985. SPE 14422. Society of Petroleum Engineers.